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This thesis will present the concept of arbitrage and some applications of arbitrage pricing. Arbitrage opportunity means that there is a possibility to make money without any initial investment and without a risk of losing money. To start, some definitions are introduced in the fields of measure theory, probability theory and mathematical finance. Then the guidelines of market models considered throughout the thesis will be defined. The mathematical definition of arbitrage and arbitrage pricing are introduced first in simple setting of one period market model and then in multi-period market model.

As a main result of this thesis are introduced and proven The fundamental theorems of arbitrage pricing. The first fundamental theorem of arbitrage pricing shows that a market is arbitrage free if and only if there exists at least one risk neutral probability measure equivalent to original probability measure such that the discounted prices are martingales with respect to this risk neutral measure. This will be proven for multi-period market model. The second fundamental theorem of arbitrage pricing shows that the completeness of a market model is equivalent to existence of unique risk-neutral probability measure. This will be proven for one period market model.

Finally, I look into some investing and hedging strategies replicating payoffs and portfolio insurance. Some examples of commonly used options strategies will be introduced such as butterfly spread and iron condor.

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On the concept of arbitrage and some applications of arbitrage pricing

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Contents

1 Introduction

The purpose of this thesis is to study the concept of arbitrage and applications of arbitrage pricing. Arbitrage opportunity means a situation where there is an opportunity to make money without a risk. An example of this would be a trader exploiting different prices for the same asset in different markets. Then the trader could buy the asset at a lower price on one market and sell it with a higher price at a other market. Trader would be making money surely without any risk of losing money. One of the main principles in security markets is that arbitrage opportunities shouldn't exist, in other words that there shouldn't be a risk free way of making money. Normally these opportunities disappear quickly from free markets.

For agents to achieve optimal portfolios, meaning a optimal collection of investments, it's crucial to exclude arbitrage from the market. Arbitrage pricing, also called as rational pricing, is an asset pricing model where prices are calculated so that they are arbitrage free with respect to equilibrium pricing. Equilibrium pricing is a market pricing system determined by supply and demand of the goods. Utility functions measure preferences over a set of goods and services and equilibrium prices exclude arbitrage opportunities when agents have strictly increasing utility functions. Utility and other terms used in this thesis are explained in more detail for example in following references: Investopedia website [15], The Optionsguide website [20] and Principles of Financial economics (Stephen F. LeRoy and Jan Werner) [2].

The construction of the thesis is the following. In Section 2, we introduce some of the key definitions, notions and results regarding measure theory, probability theory and financial economics needed to understand the concept of arbitrage and arbitrage pricing. We also outline the market model used in this thesis. In Section 3, we introduce a simplified one period market model and guidelines of arbitrage pricing. We also introduce the main theorems of the thesis, Fundamental theorems of arbitrage pricing and prove the second one for one period model. In Section 4, we introduce a multi-period market model and prove the main result of this thesis the First fundamental theorem of arbitrage pricing for multi-period model. In Section 5, we look into the ideas behind a few options trading strategies and portfolio insurance which are applications of arbitrage pricing.

2 Definitions and background theory

Risk and uncertainty are closely related to security markets and investing. Because of this we go through some of the main results of probability theory for which we need some measure theory as well. In this section, we also introduce some basic economics terminology needed in arbitrage pricing.

2.1 Measure theory

Measure theory is needed to understand probability theory. In this section we go through a few basic definitions of measure theory and define 'spaces' used in this thesis. A space is a wide concept for a set with some further structure. Crucial definition in this chapter is sigma-algebra, which are needed to define measures and to manage parts of all information given by sets. Definitions of σ-algebra and sub-σ-algebra are especially needed in this thesis to model multi-period markets for managing the partial information of sets. Multi-period market model will be introduced in Section 4.1. Definitions of compactness and convexity are needed for Lemma 2.18 that will be needed to proof the Fundamental theorems of arbitrage pricing in Sections 3 and 4.

A mathematical space is defined as a group of mathematical objects that are seen as points. These points then have different kind of relationships between each other which then determines which kind of space we have in question. Next we define some basic spaces needed in definitions of this thesis.

Definition 2.1. A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a set, $\mathcal{F}$ is a σ-algebra on Ω, and $\mathbb{P}$ is a probability measure.

Note 2.2. The definition of σ-algebra is given later on this thesis Definition 2.7 as well as definition of probability measure Definition 2.11.

Definition 2.3. A *topological space* is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a collection of open subsets of X and satisfies the following conditions:

i) The empty set $\emptyset$ and X itself belongs in τ.

ii) The intersection of a finite number of sets in τ is also in τ.

iii) The union of an arbitrary number of sets in τ is also in τ.

The elements of τ are called open sets and the collection τ is called a topology on X.

Definition 2.4. Let $T = (S, \tau)$ be a topological space. $T = (S, \tau)$ is a *Hausdorff space* if for any two distinct elements $x, y \in S$ there exist disjoint open sets $U, V \in \tau$ containing x and y respectively:

$$\forall x, y \in S,\ x \neq y,\ \exists U, V \in \tau,\ x \in U,\ y \in V \implies U \cap V = \emptyset.$$

Random variable is an object that gets its values based on outcomes of a random phenomenon, like movements in financial markets. For example future prices of financial securities are random variables.

Definition 2.5. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a measurable map from Ω to a measurable space $(\Omega', \mathcal{F}')$ is called a *random variable* (with values in Ω').

Note 2.6. Usually and in this thesis we have real-valued random variables. A real-valued random variable is a function mapping a probability space into the real line $\mathbb{R}$. A real valued random variable has a probability distribution.

In probabilistic sense a sigma-algebra is a set of results from a random experiment. Financial markets can be seen as a sequence of these random experiments. Sigma-algebras are needed for example to form conditional expectations. Expected value, and especially expected value with some added conditions, are central in forming pricing systems in financial markets. When we add conditions to expectation we speak about conditional expectation. When conditional expectation is concerned we are often interested in sets that represent only part of all the possible information that can be observed. This part of all the possible information is defined as a sub-sigma-algebra which is a subset of the original sigma-algebra. Sub-sigma-algebra includes only the information that is relevant for the part we want to view or only the information that is available to us.

Definition 2.7. Suppose Ω is a set, and $\mathcal{F} \subset 2^{\Omega}$ a collection of it's subsets. The collection $\mathcal{F}$ is called a *σ-algebra* if the following conditions are satisfied:
i) $\emptyset \in \mathcal{F}$,
ii) if $A \in \mathcal{F}$, then $A^c := \Omega / A \in \mathcal{F}$,
iii) if $A_1, A_2, ...$ is a sequence of subsets of Ω such that $A_i \in \mathcal{F}$ for all i, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Definition 2.8. A set equipped with a σ-algebra is called a *measurable space*, and the elements of the σ-algebra are called *measurable sets.*

Definition 2.9. Suppose Ω is a set, and $\mathcal{A} \subset 2^{\Omega}$ and $\mathcal{B} \subset 2^{\Omega}$ are σ-algebras on Ω. $\mathcal{B}$ is called a *sub-σ-algebra* if $\mathcal{B} \subseteq \mathcal{A}$.

Definition 2.10. Given a measurable space $(\Omega, \mathcal{F})$, a function $\mu : \mathcal{F} \to \mathbb{R}_+ \cup \{+\infty\}$ is called a *measure* if it satisfies the following properties:

i) $\mu(\emptyset) = 0$;

ii) if $A_1, A_2, ...$ is a sequence of disjoint sets, such that $A_i \in \mathcal{F}$ for all i, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. This is called a σ-additivity.

Definition 2.11. A measure μ is called a *probability measure* if $\mu(\Omega) = 1$.

Definition 2.12. We say that *two probability measures* $\mathbb{P} \sim Q$ *are equivalent* if

$$\mathbb{P}(A) = 0 \;\Leftrightarrow\; Q(A) = 0, \qquad \forall A \in \mathcal{F}.$$

Example 2.13. A *Borel set* is any set in a topological space $(X, \mathcal{T})$ that can be formed from open sets (or from closed sets) through the operations of countable union, countable intersection and relative complement. For a topological space $(X, \mathcal{T})$ the collection of all Borel sets forms a σ-algebra known as the *Borel σ-algebra.* The Borel σ-algebra on $(X, \mathcal{T})$ is the smallest σ-algebra containing all open sets (or all closed sets). Any measure defined on the Borel sets is called a *Borel measure.*

Definition 2.14. Let V be a vector space, for example $V = \mathbb{R}^d$, the euclidean space. We say that a set $A \subseteq V$ is *compact* if and only if A is closed and bounded. A set is called closed when it contains all of it's limit points. A set is called bounded when it has lower and upper bounds meaning that all its points are within a fixed distance of each other.

Definition 2.15. Let B be a Borel set of a Hausdorff topological space $T = (S, \tau)$ where $\mathcal{B}$ is a σ-algebra over B and $\tau \subset \mathcal{B}$. Let γ be a measure on $\mathcal{B}$. Let U be any open set and K compact set such that $K \subseteq U$.

i) The measure γ is called *inner regular* or *tight* if, for any open set U, $\gamma(U)$ is the supremum of $\gamma(K)$ over all compact subsets K of U:

$$\gamma(U) = \sup\{\gamma(K) \mid \text{compact } K \subseteq U\}.$$

ii) The measure γ is called *outer regular* if, for any Borel set B, $\gamma(B)$ is the infimum of $\gamma(U)$ over all open sets U containing B:

$$\gamma(B) = \inf\{\gamma(U) \mid \text{open } U \supseteq B\}.$$

iii) The measure γ is called *locally finite* if every point p of S has a neighbourhood U for which $\gamma(U)$ is finite.

$$\forall p \in S, \exists U \in \tau \text{ s.t. } p \in U \text{ and } |\gamma(U)| < +\infty.$$

The measure γ is called a *Radon measure* if it fills the conditions *i)* $-$ *iii)* and is inner regular, outer regular and locally finite.

Next we give a definition for convex sets and convex functions and introduce some useful propositions based on convexity. Concept of convexity and the separation lemma, Lemma 2.18, introduced under are relevant to proof the Fundamental theorem of arbitrage pricing Theorem 4.5.

Definition 2.16. Let V be a vector space, for example $V = \mathbb{R}^d$, the euclidean space. We say that a set $A \subseteq V$ is *convex* if and only if

$$x, y \in A,\ 0 \leq \alpha \leq 1 \Rightarrow\ (\alpha x + (1-\alpha)y) \in A.$$

Proposition 2.17. *Intersection of two convex sets is a convex set.*

Proof. Let A and W be convex sets. We want to show that $A \cap W$ is also convex. Take $x_1, x_2 \in A \cap W$, and let x lie on the line segment between these two points. Then $x \in A$ because A is convex, and similarly, $x \in W$ because W is convex. Therefore $x \in A \cap W$, as desired. □

Following Lemma 2.18 is taken from reference [3, Lemma 2.35]. Additional steps are added to the proof for the convenience of the reader.

Lemma 2.18. *Suppose $A \subset \mathbb{R}^n$ is convex and compact, and W is a vector subspace of $\mathbb{R}^n$ disjoint from A. Then there exists $z \in \mathbb{R}^n$ such that $\langle z, a\rangle > 0$ for all $a \in A$, and $\langle z, w\rangle = 0$ for all $w \in W$, where $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ is the Euclidean inner product in $\mathbb{R}^n$.*

Proof. Consider a set

$$A - W = \{a - w : a \in A,\ w \in W\},$$

where A and W are disjoint and then $0 \notin A - W$. Since A is convex and W is convex by Proposition 2.1 also $A - W$ is convex.

Donate a set $B = \{\mid x \mid : x \in A - W\}$. We will show that B is closed and consider a convergent sequence $b_n \in B,\ b_n \to b$. Then there exists $x_n \in A - W$ such that $\mid x_n \mid = b_n$, and we can write $x_n = a_n - w_n$ for some $a_n \in A,\ w_n \in W$. The sequence a_n has a convergent subsequence $a_{n_k} \to a$, since A is compact. The sequence w_{n_k} is bounded since otherwise $\mid a_{n_k} - w_{n_k} \mid = b_{n_k}$ would go to infinity. So selecting a further subsequence gives a convergent sequence $w_{n_{k_m}} \to w$. Now by continuity of $x \mapsto \mid x \mid$, we have $\mid a - w \mid = b$ so $b \in B$.

Take z to be the element of $A - W$ closest to 0, so that for any other element c of $A - W$ we have $\mid c \mid \geq \mid z \mid > 0$ (recall that $0 \notin A - W$). The existence of z follows from the fact that the set $B = \{\mid x \mid : x \in A - W\}$ is closed, does not contain 0, and so has the smallest positive element, which is the Euclidean norm of some element of $A - W$.

Consider the triangle in $\mathbb{R}^n$ with vertices $0, z, c$. We shall prove that $\langle z, c\rangle \geq \langle z, z\rangle = \mid z \mid^2 > 0$. We have

$$\langle z, c\rangle \geq \mid z \mid\mid c \mid \cos\alpha = \mid z \mid\mid c \mid \frac{\mid z \mid}{\mid y \mid} = \mid z \mid^2 \frac{\mid c \mid}{\mid y \mid}$$

so it is sufficient to show that $\mid c \mid \geq \mid y \mid$. Suppose, $\mid c \mid < \mid y \mid$. The line segment joining c and z is contained in the convex set $A - W$. But the line intersects the interior of a ball with the centre at 0 and radius $\mid z \mid$. So this segment contains a point with distance to 0 strictly smaller than $\mid z \mid -a$ contradiction with the definition of z.

This gives $\langle z, a - w\rangle = \langle z, a\rangle - \langle z, w\rangle \geq \langle z, z\rangle$ for each $w \in W$. However, W is a vector space, so $nw \in W$ for each n, which makes the inequality impossible unless $\langle z, w\rangle = 0$. This in turn yields $\langle z, a\rangle \geq \langle z, z\rangle > 0$ so both claims are proved.

□

Definition 2.19. A function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is called a *proper convex function* if $f(x) < \infty$ for some $x \in \mathbb{R}$ and if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

for $x, y \in \mathbb{R}$ and $\alpha \in [0,1]$. The *effective domain* of f, denoted by dom f, consists of all $x \in \mathbb{R}$ such that $f(x) < \infty$.

The effective domain of f, dom f, is a real interval $S =$ dom f. As a function $f : S \to \mathbb{R}$ the function f is convex in a usual sense as in Definition 2.16. Any convex function $f : S \to \mathbb{R}$ defined in some non-empty interval S can be viewed as proper convex function by defining $f(x) = +\infty$ for $x \in \mathbb{R} \setminus S$. Next we introduce a proposition of continuity and differentiability properties of a proper convex function on dom f which are later used in Section 5.2 in Example 5.10.

Proposition 2.20. *Let f be a proper convex function, and denote by D the interior of dom f.*

a) f is upper semi continuous on dom f and locally Lipschitz continuous on D.

b) f admits left- and right-hand derivatives

$$f'_-(y) = \lim_{x \uparrow y} \frac{f(x) - f(y)}{x - y} \quad \text{and} \quad f'_+(y) = \lim_{z \downarrow y} \frac{f(z) - f(y)}{z - y}$$

at each $y \in D$. Both f'_- and f'_+ are increasing functions and satisfy $f'_- \leq f'_+$.

c) The right-hand derivative f'_+ is right-continuous, the left-hand derivative f'_- is left-continuous.

d) f is differentiable a.e. in D, and for any $x_0 \in D$

$$f(x) = f(x_0) + \int_{x_0}^{x} f'_+(y)dy = f(x_0) + \int_{x_0}^{x} f'_-(y)dy, \quad x \in D. \tag{2.1}$$

Proof. We start by proving *b*). For $x, y, z \in D$ with condition $x < y < z$, we take $\alpha \in (0,1)$ such that $y = \alpha z + (1-\alpha)x$. Using the convexity of f, we get

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}.$$

Because the difference quotient

$$\frac{f(x) - f(y)}{x - y}$$

is an increasing function of x, which shows the existence of the left- and right-hand derivatives. Now we get $f'_-(y) \leq f'_+(y) \leq f'_-(z) \leq f'_+(z)$ for $y < z$.

Next we prove *a*). Let $z \in$ dom f, and take a sequence $(x_n) \subset$ dom f such that $x_n \to z$. We may assume $x_n \downarrow z$ or $x_n \uparrow z$ without loss of generality. In either case $x_n = \delta_n x_1 + (1-\delta_n)z$, where $\delta_n \downarrow 0$. Because of convexity of f holds that

$$\limsup_{n\uparrow\infty} \leq \limsup_{n\uparrow\infty}(\delta_n f(x_1) + (1-\delta_n)f(z)) = f(z),$$

and so f is upper semicontinuous. To prove local Lipschitz continuity, take $a \leq x < y \leq b$ such that $[a,b] \subset D$. From part *b*) we get

$$f'_-(a) \leq f'_+(a) \leq \frac{f(x) - f(y)}{x - y} \leq f'_-(y) \leq f'_+(b).$$

Hence f is a Lipschitz continuous on $[a,b]$ with Lipschitz constant $L :=\mid f'_+(a) \mid \vee \mid f'_-(b) \mid$.

Proof of part *c*). Continuity of f shows that for $x < z$

$$\frac{f(z) - f(x)}{z - x} = \mathop{lim}_{y\downarrow x} \frac{f(z) - f(y)}{z - y} \geq \mathop{lim\ sup}_{y\downarrow x} f'_+(y).$$

Taking $z \downarrow x$ yields $f'_+(x) \geq \mathop{lim\ sup}_{y\downarrow x} f'_+(y)$. Since f'_+ is increasing, we must in fact have $f'_+(y) \to f'_+(x)$ as $y \downarrow x$. The left-continuity of f'_- is shown in a same way.

Finally the proof of part d). Since is proven that the function f is Lipschitz continuous, it is absolutely continuous. By Lebesgues differentiation theorem, f is hence almost everywhere differentiable and equal to the integral of its derivative, which is equal to $f'_-(x) = f'_-(y)$ for almost every $x \in D$.

$\square$

2.2 Probability theory

Measure theory works as a base for probability theory and probability space was already defined in previous section in Definition 2.1. Expectation is a central concept in probability theory and it's also important while speaking of financial mathematics and pricing systems as in general the pricing is based on expectation. We introduce some of the basic properties of expectation and conditional expectation. In this section we also look into a definition of stochastic process and martingales and see some examples.

Definition 2.21. Let Ω' be a set and $\mathcal{F}'$ a σ-algebra over Ω'. If X is a random variable with values in Ω', then the measure on $\mathcal{F}'$ given by

$$\mu(A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\})$$

is called the *distribution* of random variable X.

Definition 2.22. The *cumulative distribution function* of a real valued random variable X is a function $F : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) = \mathbb{P}(X \leq x), \quad x \in \mathbb{R}.$$

Random variables can be either discrete, continuous or combination of these two. In this thesis we will concentrate on strictly discrete and strictly continuous cases. Conditional expectation is defined separately for these two cases in Definition 2.32 so next we define them.

Definition 2.23. Random variable X is

i) *discrete* when the image of X is countable. Discrete random variable has a probability mass function $p : \mathbb{R} \to [0, 1]$ such that

$$p_X(x_i) = \mathbb{P}(X = x_i),$$

where $x_i \in \mathbb{R}$, $\sum p_X(x_i) = 1$, $p(x_i) > 0$ and $p(x) = 0$ for all other x.

ii) *continuous* when the cumulative distribution function of X is absolutely continuous for all $x \in \mathbb{R}$, and a function $f : \mathbb{R} \to \mathbb{R}$, if

$$F(x) = \int_{-\infty}^{x} f(y)dy, \quad x \in \mathbb{R}.$$

Then also

$$\mathbb{P}(a \leq x \leq b) = \int_{a}^{b} f(y)dy, \quad a, b \in \mathbb{R}, \ a < b.$$

If the cumulative distribution function of continuous random variable is differentiable almost everywhere we can form the probability density function by calculating the derivative of the cumulative distribution function

$$F'(x) = \frac{d}{dx}F(x) = f(x).$$

If we have two random variables X and Y that are defined on a same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ they form a random vector $(X, Y) : \Omega \to \mathbb{R}^2$ and there exists a joint probability distribution for a random vector (X, Y). The joint probability distribution has a joint cumulative distribution function and joint probability mass function (discrete random variables) or joint probability density function (continuous random variables).

Definition 2.24. Probability distribution function of random vector (X, Y) meaning a random variable with values in Ω', then the measure on $\mathcal{F}'$ given by

$$\mathbb{P}((X, Y) \in A) = \mathbb{P}(\{\omega \in \Omega : (X(\omega), Y(\omega)) \in A\})$$

is called a *joint probability distribution* of the random variables X and Y.

Definition 2.25. Let (X, Y) be a random vector. Then function $F : \mathbb{R}^2 \to \mathbb{R}$

$$F(x, y) = P(X \leq x, Y \leq y), \quad x, y \in \mathbb{R},$$

is the cumulative distribution function of (X, Y) meaning that is the *joint cumulative distribution function* of random variables X and Y.

Definition 2.26. Random vector (X, Y) has a *joint density function* such that

$$i) \ p_{X,Y}(x, y) = \mathbb{P}(X = x \text{ and } Y = y), \text{ when } X, Y \text{ is discrete.}$$
$$ii) \ f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y}, \text{ when } X \text{ is continuous.}$$

Definition 2.27. Assume random vector (X, Y) has a continuous probability distribution with probability density function $f_{X,Y}$. Then the random variable X has a *conditional density function* in respect to given $Y = y$ such that

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \quad x \in \mathbb{R},$$

when $f_Y(y) > 0$.

Next we introduce Fubini's theorem which is needed for an application of arbitrage pricing called a portfolio insurance which is introduced later in this thesis in Section 5.2.

Theorem 2.28. Fubinis theorem *implies that two iterated integrals are equal to the corresponding double integral across its integrands. As a consequence it allows the order of integrations to be changed in iterated integrals. Suppose X and Y are complete measure spaces. Suppose $f(x, y)$ is $X \times Y$ measurable. If*

$$\int_{X \times Y} \mid f(x, y) \mid d(x, y) < \infty,$$

where the integral is taken with respect to a product measure on the space over $X \times Y$, then

$$\int_X \left(\int_Y f(x, y)\, dy \right) dx = \int_Y \left(\int_X f(x, y)\, dx \right) dy = \int_{X \times Y} f(x, y)\, d(x, y), \tag{2.2}$$

where the first two integrals are iterated integrals with respect to two measures and the third is an integral with respect to a product of these two measures. Fubini's theorem is proven for example in reference [18].

The expected value of a random variable is the integral of the random variable with respect to it's probability measure.

Definition 2.29. If X is a random variable defined in probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the *expected value* is defined as a Lebesgue integral over measure space:

$$\mathbb{E}(X) = \int_\Omega X(w) d\mathbb{P}(w).$$

Previous definition is for a general case and valid for all random variable X for which the expected value exists, that is when $\mathbb{E}(\mid X \mid) < \infty$. When we have information about continuity of the random variable we can define expectation more practical.

Definition 2.30. If X is a random variable defined in probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the expected value is defined as

$$i)\ \mathbb{E}(X) = \sum_{x \in \Omega} xp(x), \text{ when } X \text{ is discrete.}$$
$$ii)\ \mathbb{E}(X) = \int_{\mathbb{R}} xf(x)dx, \text{ when } X \text{ is continuous.}$$

Conditional expectation is needed for martingale Definition 2.40. Expectation in common sense is a value that represents the average of a large number of independent realizations of the random variable over a large number of occurrences. Conditional expectation is the same with given conditions that are known to occur.

Definition 2.31. Elementary *conditional expectation* for random variable X given an event A is the expectation with respect to original probability measure and is defined as

$$\mathbb{E}(X \mid A) = \frac{\mathbb{E}(X \mathbb{1}_A)}{\mathbb{P}(A)}.$$

Definition 2.32. Let Y be a scalar random variable and X be a random variable with values in a finite or countable space. Then the conditional expectation is defined

$$i)\ \mathbb{E}(X \mid Y)(y) = \mathbb{E}(X \mid Y = y) = \sum_{x \in \Omega} x f_{X|Y}(x \mid y), \text{ when } X \text{ is discrete.}$$
$$ii)\ \mathbb{E}(X \mid Y)(y) = \mathbb{E}(X \mid Y = y) = \int_{\mathbb{R}} x f_{X|Y}(x \mid y)dx, \text{ when } X \text{ is continuous.}$$

Next we introduce some of the basic properties of conditional expectation which are proven for example in book Foundations of the Theory of Probability reference [10, Chapter IV].

Definition 2.33. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, and let X be random variable in probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that the random variable ξ is the conditional expectation of X with respect to $\mathcal{G}$, and denote it by $\mathbb{E}[X \mid \mathcal{G}]$, if following holds

1. ξ is a random variable in probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
2. ξ is $\mathcal{G}$-measurable,
3. $\mathbb{E}[\xi \mathbb{1}_A] = \mathbb{E}[X \mathbb{1}_A]$, for all $A \in \mathcal{G}$.

Theorem 2.34. *Suppose X and Y are random variables in probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra. Then the conditional expectation satisfies the following properties:*

a) $\mathbb{E}(\alpha X + \beta Y \mid \mathcal{G}) = \alpha \mathbb{E}(X \mathcal{G}) + \beta \mathbb{E}(Y \mathcal{G})$ *almost surely.*

b) If $X \geq 0$ *almost surely, then* $\mathbb{E}(X \mid \mathcal{G}) \geq 0$ *almost surely.*

c) If $X \perp\!\!\!\perp Y$, *then* $\mathbb{E}(X \mid Y) = \mathbb{E}(X)$.

d) $\mathbb{E}(\mathbb{E}(X \mid Y)) = \mathbb{E}(X)$.

e) If X *is* $\mathcal{G}$*-measurable, meaning that* $X \in \mathcal{G}$, *then* $\mathbb{E}(\mathbb{E}(X \mid \mathcal{G})) = X$.

f) If X *is* $\mathcal{G}$*-measurable, meaning that* $Y \in \mathcal{G}$, *then* $\mathbb{E}(\mathbb{E}(XY \mid \mathcal{G})) = Y\mathbb{E}(X \mid \mathcal{G})$.

g) If $X_i \geq 0$, $X_i \to X$ *almost surely,* $\mathbb{E}(\mathbb{E}(X_i \mid \mathcal{G}))$ *exists for all* i, *and* $\mathbb{E}(\mid X \mid) < \infty$, *then* $\mathbb{E}(X \mid \mathcal{G})$ *exists and is equal to* $lim_{i \to \infty} \mathbb{E}(X \mid \mathcal{G})$ *almost surely.*

A stochastic process is a collection of random variables indexed by a variable, usually t representing time. They are a direct generalization of random vectors and are used in finance to model at least seemingly random changes. According to for example Paul-André Meyer [9] stochastic processes and the theory has been around since about 1950. In multi-period market model the price fluctuations are described as stochastic processes in discrete time.

Definition 2.35. For a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable space (S, Σ) a *stochastic process* is a collection of S-valued random variables which can be written as:

$$\{X(t) : t \in T\}.$$

Example 2.36. *A random walk* is one example of a mathematical object that is a stochastic process. Random walk consists of random steps that form a path on some mathematical space. A very simple example of this is the random walk on the integer number line, $\mathbb{N}$. The random walk starts at 0 and at each step moves with equal probability either $+1$ or -1. Other examples more relevant to this thesis are the price of a fluctuating stock and the financial situation of a gambler. Both can be described as random walks. In reality they are not often truly random but can be approximated by random walk models.

Example 2.37. *A Markov chain* is a stochastic process that satisfies the Markov property. We say that a stochastic process has the Markov property if the future states conditional probability is not anyhow dependent on events happened previously but only of current state. Markov property is also called descriptively as memoryless property. An easy example of this is a hat containing five balls, two red and three yellow. If we draw one ball and know the colour of it, this effects on the probability distribution of the next ball drawn if we do not put it back before the next draw. If we do put the drawn ball back before the second draw the previous draw does not effect on the probability distribution of the next draw. In this case we would have the Markov property.

Pricing and existence of arbitrage opportunities depend on equivalent martingale measures which we will look closer into later on this thesis in Sections 3.2 and 4.1. We now introduce the probability theory definitions of filtered probability space and martingale so on next section we can go through the definition of equivalent martingale measure. Definitions of filtration and adapted and predicted processes are needed for martingales and for the proof of Theorem 4.5. We can think of filtration as a flow of information. The σ-algebra $\mathcal{F}_n$ contains the events that can happen until time n. An adapted process is a process that can not look into the future. The predictable process means that the value of a process at time n is already known at time $n-1$ [14].

Definition 2.38. *Filtration* is an increasing sequence of σ-algebras such that

$$\mathcal{F}_0 \in \mathcal{F}_1 \in \cdots \in \mathcal{F}_T.$$

A probability space filled with filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ldots,T}, \mathbb{P})$ is called a *filtered probability space*.

Definition 2.39. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t)_{t=0,\ldots,T}$ be a filtration. Then

a) a discrete time stochastic process $X_0, X_1, \ldots$ is called *adapted* to a filtration $\{\mathcal{F}_n\}$ if X_n is $\mathcal{F}_n$ measurable for all $n = 0, 1, \ldots$

b) a discrete time stochastic process $X_1, X_2, \ldots$ is called *predictable* with respect to a filtration $\{\mathcal{F}_n\}$ if X_n is $\mathcal{F}_{n-1}$ measurable for all $n = 0, 1, \ldots$

A martingale is a sequence of random process. For sequence called martingale it's present value is the conditional expectation for the next value. Martingales are central in the mathematics of randomness. They appear in the general theory of stochastic processes, in the algorithmic theory of randomness and in mathematical statistics. The concept and word martingale came from betting strategies for gambling in 18th century France. This was right in the beginning of probability theory but martingales came much more important when the stochastic processes became central while studying randomness in mathematics and science in general. More detailed history of martingales in The Splendors and Miseries of Martingales reference [11].

Definition 2.40. Let X_n be a stochastic prcess adapted to a filtration $\{\mathcal{F}_n\}$, such that $\mathbb{E}|X_n| < \infty$ for all n. Then we can say that *i)* X_n is a *submartingale* if $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq X_n$ almost surely for all $n = 0, 1, \ldots$

ii) X_n is a *supermartingale* if $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq X_n$ almost surely for all $n = 0, 1, \ldots$

iii) X_n is a *martingale* if $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ almost surely for all $n = 0, 1, \ldots$

Example 2.41. An example of a martingale would be an *unbiased random walk*. The walk is said to be unbiased if the value has the same probability to move up or down.

Example 2.42. Suppose X_n is a gamblers assets after n tosses of a fair coin. In the game the gambler wins 1 euro if the coin comes up heads and loses 1 euro if it comes up tails. The gamblers conditional expected assets after the next trial is equal to their present asset amount so this sequence is a martingale.

Example 2.43. *De Moivres martingal.* Suppose a gambler tosses an unfair coin with probability p of coming up heads and probability $q = 1 - p$ of coming tails. Let $X_{n+1} = X_n \pm 1$. When heads come up this is $+$ and when tails comes up $-$. Let

$$Y_n = \left(\frac{q}{p}\right)^{X_n}.$$

Now

$$\begin{aligned}
\mathbb{E}[Y_{n+1} \mid X_1, \ldots, X_n] &= p \cdot \left(\frac{q}{p}\right)^{X_n+1} + q \cdot \left(\frac{q}{p}\right)^{X_n-1} \\
&= p \cdot \left(\frac{q}{p}\right) \cdot \left(\frac{q}{p}\right)^{X_n} + q \cdot \left(\frac{p}{q}\right) \cdot \left(\frac{q}{p}\right)^{X_n} \\
&= q \cdot \left(\frac{q}{p}\right)^{X_n} + p \cdot \left(\frac{q}{p}\right)^{X_n} = \left(\frac{q}{p}\right)^{X_n} = Y_n.
\end{aligned}$$

So Y_n is a martingale with respect to X_n.

2.3 Mathematical definitions in finance

In this section we introduce the guidelines of market models considered in this thesis and some of the financial economics mathematical notations and definitions needed in arbitrage pricing.

The market setting. Throughout the thesis we will be looking into simplified setting of real markets and suppose that all actions, meaning buying and selling, take place on discrete time points $k = 0, 1, \dots, T$. We consider there to be $1, \dots, N$ assets. The price of security n at time k is denoted by

$$S_n(k), \;\; n \in \{1, \dots, N\}.$$

The price at time 0, $S_n(0)$, is a known constant and the price at time $k > 0$, $S_n(k)$, is a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Elements $\omega_m = \{\omega_1, \dots, \omega_M\} \in \Omega$ are defined as the states of the market and they implicate the possible future scenarios. For the simplicity of proofs in this thesis the amount of future scenarios is finite, that is, $m \in \{1, \dots, M\}$ for a fixed M in N. It's good to notice that in real market setting we can't normally know the amount of all possible scenarios but most theorems and lemmas introduced in this thesis can be generalized for a sample space Ω with infinitely many outcomes. When ω_m is known all the prices of all securities $S_n(k, \omega_m)$ are known. An *interest rate* is the amount of interest paid of a loan during a period of time, usually one year. Deterministic interest rate is denoted by $i_j \geq 0$, which here is a constant responding to time intervals $j = 1, 2, ...$ for the simplicity even though in a real market this is not often the case. This means if a deposit C is made at time $(j-1)$ for time Δ, one can withdraw at time $j\Delta$ the amount

$$C(1 + i_j)^{\Delta}, \text{ where } j = 1, 2, \dots.$$

Rate of return is a profit on an investment during a period of time, usually one year. *Return* denoted by r_n is the made profit of an investment and it is a random variable defined as

$$r_n = \frac{S_n(k) - S_n(k-1)}{S_n(k-1)}, \qquad k = 1, 2, \dots,$$

which determines the price $S_n(k) = S_n(k-1)(1 + r_n)$.

At time 0 an agent obtains a *portfolio* which is denoted by

$$\theta = (\theta_1, \dots, \theta_N)^T \in \mathbb{R}^N,$$

where θ_n is the amount of securities n in the portfolio. In financial economics portfolio means any collection of financial derivatives such as investments, stocks, bonds and cash. Optimizing the portfolio means finding the *optimal portfolio* which is in some way the best collection of these assets. Usually this means maximized expected return and minimized risk. When the agent chooses a portfolio the future prices are unknown. The value of a portfolio in the future is a random variable and at time k the portfolio can be sold for the price that is denoted by $S(k, w_m)\theta$, where w_m represents the happened event in markets.

We assume that there are shares and derivatives of shares on the market of which value depends on the value of underlying shares in some way. We assume that there is a risk-free security on the market such as a k-year *zero coupon bond.* We assume further that the security 1 is a k-year zero-coupon bond, that is $S_1(k) = 1$ and $S_1(0) = \frac{1}{(1+r)^k}$ almost surely. Common derivatives are call options, put options, future, guaranteed interest rate and currency option. Call options give their owner the right but not obligation to buy an underlying share at a fixed strike price on the expiration date of the option. Similarly put options give their owner the right but not obligation to sell an underlying share at a fixed strike price on the expiration date of the option. Future is an obligation to buy or sell the share on a fixed date at a fixed price. Guaranteed interest rate means that an investment is guaranteed to earn minimum certain amount of interest agreed in advance in certain time. Currency option means that the owner has the right but not obligation to exchange certain currency on another one at a fixed rate on expiration date.

Example 2.44. The owner of a call option on the i:th asset has the right but not the obligation to buy the i:th asset at agreed time for a fixed price K, called a *strike price.* The corresponding payoff C^{call} is given by

$$C^{call} = (S^i - K)^+ = \begin{cases} S^i - K & \text{if } S^i > K, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Similarly the owner of a put option on the i:th asset has the right but not the obligation to buy the i:th asset at agreed time for a fixed price K, called a *strike price.* The corresponding payoff C^{put} is given by

$$C^{put} = (K - S^i)^+ = \begin{cases} K - S^i & \text{if } S^i < K, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Existence of *equivalent martingale measure* is a condition for arbitrage pricing according to a central result of this thesis Fundamental theorem of arbitrage pricing introduced and proven in Section 4.5. Equivalent martingale measure is also called risk neutral probability measure and equilibrium measure. According to Investopedia [15, keyword equivalent martingale measure]: "Equivalent Martingale Measures is a probability distribution of expected payouts from an investment used in asset pricing. The distribution is adjusted for the risk premiums of investors as a whole. An equivalent martingale measure thus factors in the average investor's degree of risk aversion to allow for a more straightforward calculation of the present value of a security". Equivalent martingale measure and the existence of it are in key position in Fundamental theorems of arbitrage pricing, Theorems 3.4 and 3.5, and that is why we define it in following two Definitions 2.45 and

2.46. First fundamental theorem of arbitrage pricing will be proven for multi-period market model in Section 4.5 and the Second fundamental theorem of arbitrage pricing will be proven for one period market model in Section 3.2.

Definition 2.45. The finite sequence $Q^1 = \{q_1, \dots, q_M\}$ of positive numbers is a *equivalent martingale measure* in one period market model for $\Omega = \{\omega_1, \dots, \omega_M\}$ and stock prices $\{S_n(0), S_n(1) : n = 1, \dots, N\}$ if $\sum_{j=1}^{M} q_j = 1$ and $\mathbb{E}_Q(S_n(1)) = (1+i)S_n(0)$ for all n, where $\mathbb{E}_Q$ denotes the expectation with respect to the probability $Q(w_j) = q_j > 0$ for $j \leq M$, and i denotes the riskless interest rate.

Definition 2.46. In one period model a probability measure Q on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *equivalent martingale measure* if

$$S_n(0) = \mathbb{E}_Q\left[\frac{S_n(1)}{1+r}\right], \quad n = 0, 1, \dots, N.$$

Similarly for multi period model a probability measure Q on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *equivalent martingale measure* if

$$S_n(0) = \mathbb{E}_Q\left[\frac{S_n(T)}{(1+r_1)\cdots(1+r_T)}\right], \quad n = 0, 1, \dots, N.$$

Utility is a term of economics that was brought into use for the first time in 18th century by Daniel Bernoulli. Utility is the contentment combined in total of consuming a good or a service. Therefore utility will directly influence on the demand of a good or a service so it will directly influence on the price as well. According to Investopedias article reference [15, keyword utility] consumer behaviour theories assume that consumers will strive to maximize their utility which can be easily believed by common sense. *Utility function* measures which goods or services costumers prefer in terms of welfare or satisfaction. In pricing utility function measures the benefit obtained from the wealth [17]. Utility functions are used in portfolio selection problem and they are marked as u in this thesis.

Complete market is a market where transactions can be made freely without additional costs and there is a price for every asset existing. From a view of probability theory there exists an unique martingale measure for a complete market but for an incomplete market there is no such a unique martingale measure. We can understand this in a way that even though options are not risk free, in complete markets we don't need to determine what is our own risk aversion degree to price them. In complete market there exists an equilibrium price for each asset in every possible state. Definition of complete market is needed for the Second fundamental theorem of arbitrage pricing which suggests that a arbitrage free market is complete if and only if there exists exactly one risk neutral probability measure. This Theorem 3.5 is proven for one period model later on the thesis. First we define attainability which is needed for definition of complete market.

Definition 2.47. A random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is *attainable*, if there exists a self-financing strategy $\{\theta(k) \mid k = 1, \dots, T\}$ such that

$$X(T, \omega) = S(T, \omega)\theta(T) \text{ almost surely,}$$

where $X(T, \omega)$ is the value of a new financial security at time T.

Definition 2.48. Market is *complete*, if every random variable X in probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is attainable.

Note 2.49. Only few market models are actually complete in real life.

Theorem 2.50. *For $X(k)$ to be a martingale it is sufficient that for each predictable process $\theta(k)$ holds*

$$\mathbb{E}\left(\sum_{k=1}^{T} \theta(k)\Delta X(k)\right) = 0,$$

where $\Delta X(k) = X(k) - X(k-1)$.

Proof. We will show that

$$\mathbb{E}(X(k+1) \mid \mathcal{F}_n) = X(k).$$

Take $A \in \mathcal{F}_k$ and define $\theta(k) = \mathbb{1}_A$ when $k = n+1$ and $\theta(k) = 0$ otherwise. Now we can reduce the sum

$$\sum_{k=1}^{T} \theta(k)\Delta X(k) = \mathbb{1}_A \Delta X(n+1). \tag{2.5}$$

By the assumption

$$\mathbb{E}\left(\sum_{k=1}^{T} \theta(k)\Delta X(k)\right) = \mathbb{E}(\mathbb{1}_A \Delta X(n+1)) = \mathbb{E}(\mathbb{1}_A[X(n+1) - X(n)] = 0$$

so the definition of conditional expectation completes the proof. □

3 Arbitrage in the one period market model

3.1 One period market model

First we look into a simple one period market model. The setting in the one period market model is following.

Consider the market setting presented in Section 2.3. Recall probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume there is only a single trading period from the present time $k = 0$ to a fixed future time $k = T$. Here to make it even simpler we assume $T = 1$. There are N amount of assets with initial prices $S(0) = (S_1(0), \dots, S_N(0)) \in \mathbb{R}^N$. Assume security 1 is a one year zero-coupon bond meaning that its owner receives the amount 1 at time 1 yielding $S_1(0) = \frac{1}{1+r}$ and $S_1(1) = 1$. Prices $S(1) = (S_1(1), \dots, S_N(1))$ are unknown and depend on the happening scenario. Denote price process $S = \{S(0), S(1, w)\}$. We consider M different scenarios, $\omega = \{\omega_1, \dots, \omega_M\}$.

At time 0 an agent can obtain a portfolio $\theta = (\theta_1, \dots, \theta_N)^T \in \mathbb{R}^N$. The prices at time $k = 0$ are known and the prices at time $k = 1$ are random variables on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Price of a portfolio at time $k = 0$ is

$$S(0)\theta = \sum_{n=1}^{N} S_n(0)\theta_n. \tag{3.1}$$

Price of a portfolio at time $k = 1$ is

$$S(1)\theta = \sum_{n=1}^{N} S_n(1)\theta_n. \tag{3.2}$$

Wealth at time 1 is

$$V(1) = S(1)\theta. \tag{3.3}$$

Definition 3.1. Suppose an asset X has a price $X(1)$ at time 1. Its *discounted value* to time 0 is given by

$$\tilde{X}(1) = \frac{X(1)}{(1+r)} = X(0).$$

For the stock S_n we have discounted value

$$\tilde{S}_n(1) = \frac{S_n(1)}{(1+r)} = S(0)$$

and define the *discounted price process* $\tilde{S}_n = \{\tilde{S}_n(0), \tilde{S}_n(1)\} = \{S_n(0), \tilde{S}_n(1)\}$. Similarly the *discounted value process* is defined $\tilde{V} = \{\tilde{V}(0), \tilde{V}(1)\}$. Denote $\theta^* = (\theta_2, \ldots, \theta_N)$. Now

$$
\begin{aligned}
\tilde{V}(1) &= \frac{V(1)}{(1+r)} = \theta_1 \tilde{S}_1(1) + \theta^* \tilde{S}(1) = \theta_1 \frac{1}{1+r} + \theta^* \tilde{S}(1), \text{ and} \\
\tilde{V}(0) &= V(0) = \theta_1 \frac{1}{1+r} + \theta^* \tilde{S}(0),
\end{aligned} \tag{3.4}
$$

yielding

$$
\tilde{V}(1) - \tilde{V}(0) = \left(\frac{1}{1+r}\theta_1 + \theta^* \tilde{S}(1)\right) - \left(\frac{1}{1+r}\theta_1 + \theta^* \tilde{S}(0)\right) = \theta^*(\tilde{S}(1) - \tilde{S}(0))
$$

showing that the change of the discounted portfolio value results directly from the change of the discounted value of risky assets.

3.2 Arbitrage

A strong arbitrage is a portfolio with positive payoff and strictly negative price. An arbitrage is a portfolio that is either a strong arbitrage or has a strictly positive payoff and zero price. In general there can't exist a long-term arbitrage opportunity in markets. This would mean that someone could make money without risk which would lead to someone else certainly losing money which doesn't make sense. The existence of arbitrage opportunity depends on pricing. Market is *arbitrage free* if arbirtage opportunity doesn't exist.

Definition 3.2. There exists an *arbitrage opportunity* if there is a portfolio θ such that
i) $S(0)\theta \leq 0$,
ii) $S(1)\theta \geq 0$ almost surely
iii) and $\mathbb{P}(S(1)\theta > 0) > 0$.

Lemma 3.3. *The following statements are equivalent.*

a) The market model admits an arbitrage opportunity.

b) Denote $\theta^* = (\theta_2, \ldots, \theta_N)$ and $S^*(k) = (S_2(k), \ldots, S_N(k))$. There is a vector $\theta^* \in \mathbb{R}^N$ such that

$$
\theta^* \cdot S^*(1) \geq \theta^* \cdot (1+r)S^*(0) \textit{ almost surely and } P[\theta^* \cdot S^*(1) > \theta^* \cdot (1+r)S^*(0)] > 0.
$$

Proof. First we assume that $a)$ holds and and let θ^* be an arbitrage opportunity. Then $0 \geq \theta \cdot S(0) = \theta_1 \cdot \frac{1}{1+r} + \theta^* \cdot S^*(0)$ because the security 1 is a zero-coupon bond and $S_1(0) = \frac{1}{1+r}$. So $-\theta^* \cdot S^*(0) \geq \theta_1 \cdot \frac{1}{1+r} \Leftrightarrow -\theta^*(1+i)S^*(0) \geq \theta_1$ Hence

$$\theta^* \cdot S^*(1) - \theta^* \cdot (1+r)S^*(0) \geq \theta^* \cdot S^*(1) + \theta_1 = \theta^* \cdot S^*(1) + S_1(1)\theta_1 = \theta \cdot S(1).$$

Since $\theta \cdot S(1)$ is almost surely non-negative and with positive probability strictly positive it also holds that $\theta^* \cdot S^*(1) \geq \theta^* \cdot (1+r)S^*(0)$ almost surely and $P[\theta^* \cdot S^*(1) > \theta^* \cdot (1+r)S^*(0)] > 0$.

Then we assume that $b)$ holds. We claim that portfolio θ with $\theta_1 = -\theta^* \cdot S^*(0)$ is an arbitrage opportunity. By Definition 3.2 and equations 3.4: $\theta \cdot S(0) = \theta_1 S_1(0) + \theta^* \cdot S^*(0) = \theta_1 \cdot \frac{1}{1+r} + \theta^* \cdot S^*(0) = 0 \Leftrightarrow \theta_1 \cdot \frac{1}{1+r} = -\theta^* \cdot S^*(0) \Leftrightarrow \theta_1 = -\theta^* \cdot (1+r)S^*(0)$. Moreover $\theta \cdot S(1) = \theta_1 \cdot S_1(1) + \theta^* \cdot S^*(1) = -\theta^* \cdot (1+r)S^*(0) + \theta^* \cdot S^*(1)$ which is almost surely non-negative and with positive probability strictly positive which is the Definition 3.2 of arbitrage.

□

Theorem 3.4. The first fundamental theorem of arbitrage pricing. *A market is arbitrage free if and only if there exists at least one risk neutral probability measure Q that is equivalent with the original probability measure $\mathbb{P}$, such that all discounted price processes $\tilde{S}_n(k),\ k = 1, \ldots, T$, are martingales with respect to Q.*

The first fundamental theorem of arbitrage pricing will be proven for multi-period market in Section 4.1.

Theorem 3.5. The second fundamental theorem of arbitrage pricing. *An arbitrage free market model is complete if and only if there exists exactly one risk-neutral probability measure $Q \sim \mathbb{P}$, where $Q \sim \mathbb{P}$ defined as in Definition 2.12.*

Proof. Suppose that a market is arbitrage free and complete. The market is arbitrage free if and only if there exists at least one risk neutral probability measure. This is The first fundamental theorem of pricing Theorem 3.4. Suppose there exists two risk-neutral probability measures $Q_1 \neq Q_2$. Without loss of generality we may assume a finite probability space and that $\omega \in \mathcal{F}$ and $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$. Define a derivative security with a payoff

$$X(\omega) = \mathbb{1}_A.$$

By completeness Definition 2.48 there exists a portfolio $\theta \in \mathbb{R}^N$ such that $X(\omega) = S(1,\omega)\theta$ almost surely so there exists a self-financing predictable strategy such that

$$S(1,\omega)\theta = \mathbb{1}_A.$$

Using the properties of risk-neutral measures Definitions 2.45 and 2.46

$$\mathbb{E}_{Q_j}(S(1,\omega)\theta) = \mathbb{E}(\mathbb{1}_A) = Q_j(A), \quad j = 1, 2.$$

But now the discounted values

$$\frac{\mathbb{E}_{Q_j}(S(1,\omega)\theta)}{1+r} = \mathbb{E}_{Q_i}(\tilde{S}(1,\omega)\theta) = S(0)\theta,$$

where the right hand side does not depend on i so the price of the portfolio does not depend of the choice of Q so there must be only one risk neutral probability measure and $Q_1 = Q_2$.

Then we assume there exists only one risk-neutral probability measure $Q \sim \mathbb{P}$. By The first fundamental theorem of pricing market is arbitrage free. We suppose that there exists a derivative security D with payoff $X(\omega)$ that can't be replicated. Consider a set $W = S(1,\omega)\theta$. Because D was defined so that it's not self financing $D \notin W$. Vector W is a subspace of $\mathbb{R}^M$ where $\mathbb{R}^M$ is the set of all random variables on $\Omega = \{\omega_1, \ldots, \omega_M\}$. If $\theta^* = 0$ and $\theta_1 = \frac{1}{1+r}$ then $S(1,\omega)\theta = 1$ for all scenarios.

Equip $\mathbb{R}^M$ with the inner product $\langle w, v\rangle_Q = \sum_{i=1}^M w_i v_i W(\omega_i)$ and use a variant of Separation lemma, Lemma 2.18, with the subspace W and compact convex set $A = \{D\}$. From Lemma 2.18 we obtain a vector $Z = (z_1, \ldots, z_M)$ such that $\langle Z, D\rangle_Q > 0$ and $\langle Z, V\rangle_Q = 0$ for $V \in W$. The inequality implies that $Z \neq 0$. If we take $V = 1$ we have $\langle Z, 1\rangle_Q = \sum_{i=1}^n z_i Q(\omega_i) = 0$ hence $\mathbb{E}_Q(Z) = 0$, where the expectation is taken under Q.

Now define a new measure Q_1, which we hope will turn out to be a martingale probability different from Q

$$Q_1(\omega_i) = \left(1 + \frac{z_i}{2a} Q(\omega_i)\right),$$

where $a = \max_{\omega_i \in \Omega} |z_j| > 0$. This is a probability measure since

$$\sum_{i=1}^M Q_1(\omega_i) = \sum_{i=1}^M Q_(\omega_i) + \sum_{i=1}^M \left(1 + \frac{z_i}{2a} Q(\omega_i)\right) = \frac{1}{2a}\mathbb{E}_Q(Z) = 1.$$

Clearly $-a \leq z_i$ for all i, so the number $1 + \frac{z_i}{2a}$ is always positive and for at least one coordinate it is different from 1 since Z is non-zero, thus $Q_1 \neq Q$.

Now we check the martingale probability than of each S_i under Q_1.

$$\mathbb{E}_{Q_1}(S_n(1)) = \sum_{i=1}^M S_n(1,\omega_i)\left(1 + \frac{z_i}{2a} Q(\omega_i)\right) = \mathbb{E}_Q(S_n(1)) + \frac{1}{2a}\sum_{i=1}^M z_i S_n(1,\omega_i) Q(\omega_i),$$

and if we take $\theta_n = 1$ and $\theta_k = 0$ for all $n \neq k$ then $S_n(1)$ belongs to W and with $V = (v_i)_{i \leq M}$ where $v_i = S_n(1, \omega_i)$ then $\frac{1}{2a} \sum_{i=1}^{M} z_j S_n(1, \omega_i) Q(\omega_i) = \frac{1}{2a} \langle Z, V \rangle_Q = 0$ so the equation above equals to

$$\mathbb{E}_Q(S_n(1)) = S_n(0) \frac{1}{1+r},$$

so we have a replication which is a contradiction. $\square$

4 Arbitrage in the multi-period market model

4.1 Multi-period market model

Consider the market setting presented in Section 2.3. Recall probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A multi-period model can be seen as a sequence of simplified one period models presented in Section 3.1 with multiple time points $k = 0, 1, \dots, T$. The setting in multi-period market model is the following.

We assume that the transactions can be made at times $0, 1, \dots, T-1$. At time T the portfolio is realised, meaning that its value is measured by selling all the securities. There are N assets with initial prices $\mathrm{S}(0) = (S_1(0), \dots, S_N(0)) \in \mathbb{R}^N$. Assume security 1 is a j year zero-coupon bond meaning that it's owner receives the amount 1 at time j yielding $S_1(0) = \frac{1}{(1+r)^j}$ and $S_1(j) = 1$. Prices $S(k) = (S_2(k), \dots, S_N(k))$ are unknown and depend on the realized scenario. Denote price process $S = \{S(0), S(1,w), \dots, S(T,\omega)\}$. We consider M amount scenarios, $\omega = \{\omega_1, \dots, \omega_M\}$.

Trading strategy is a series of portfolios introduced in Section 3.1. At each step agents decide on how to adjust the previous portfolio depends on the information available at that current time. The strategy is a collection of random variables. At each time $k = 0, 1, \dots, T-1$ the investor knows the prices of securities at all times $0, 1, \dots, k$. The investor might have some additional information too that affects on the future decision making. Decisions to buy and sell are made based on all the available information. This situation is mathematically modelled by using sub-sigma-algebras of $\mathcal{F}$ defined in Definition 2.9. At time k the available interesting information is modelled by sigma-algebra $\mathcal{F}_k$. It is assumed that

$$\begin{aligned} &\mathcal{F}_0 = \{\emptyset, \Omega\}, \ \mathcal{F}_T = \mathcal{F} \text{ and} \\ &\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_T. \end{aligned}$$

In multi-period market model a process $\{\theta(k) \ \mid k = 1, \dots, T\}$, where

$$\theta(k) = (\theta_1(k), \dots, \theta_n(k)), \tag{4.1}$$

is called a *trading strategy*, if

$$\theta_n(k) \text{ is } \mathcal{F}_{k-1} \text{ measurable, where } k = 1, \dots, T \text{ and } n = 1, \dots, N.$$

Recall definitions of θ^* and S^* from Lemma 3.3 part b). A strategy $\theta(k)$ produces a *value process* $V(k)$ which is determined by

$$V(k) = S(k)\theta(k) = \sum_{n=1}^{N} S_n(k)\theta_n(k) = \theta_1(k+1) + S^*(k)\theta^*(k+1), \quad k = 0, 1, \dots, T,$$

where we require that

$$V(T-1) = \theta_1(T) + S^*(T)\theta^*(T).$$

There are some useful theorems regarding to self-financing portfolios. Portfolio is self-financing if no money is withdrawn or deposited after the initial forming of the portfolio. This means that all the new assets bought are financed by selling the old ones. Next we give a definition for self-financing portfolio and introduce concepts of cumulative cost and gains process.

Definition 4.1. A strategy $\theta(k)$ is called *self-financing*, if

$$V(k) = S(k)\theta(k) = S(k)\theta(k+1)$$

for all $k = 1, \ldots, T-1$. Equally for any sequence $S(k)$ denote by $\Delta S(k)$ the vector $S(k) - S(k-1)$ for $k = 1, \ldots, T$. Now we can write the self-financing condition in a form which will be useful later

$$\Delta V(k) = \Delta S(k)\theta(k) = \Delta(S(k)\theta(k+1)).$$

Similarly to discounted values in one period model Definition 3.1 we now define discounted values for multi-period model.

Definition 4.2. Suppose a stock has a price $S_n(k)$ at time k. Its *discounted value* to time 0 is given by

$$\tilde{S}_n(k) = \frac{S_n(k)}{(1+r)^k}.$$

For the value V we have discounted value

$$\tilde{V}(k) = \frac{S(k)}{(1+r)^k}$$

and define the *discounted price process* $\tilde{S} = \{\tilde{S}(0), \ldots, \tilde{S}(k)\}$. Similarly the *discounted value process* $\tilde{V} = \{\tilde{V}(0), \ldots, \tilde{V}(k)\}$.

Cumulative cost $C(k)$, $k = 0, 1, \ldots, T$, describes the total cumulative costs added together and is determined by

$C(0) = V(0)$,

$C(k) = C(k-1) + \theta_1(k+1) + S^*(k)\theta^*(k+1) - (\theta_1(k) + S^*(k)\theta^*(k))$, where $S^*(k)$ and $\theta^*(k)$ defined as in Lemma 3.3 $\theta^*(k) = (\theta_2(k), \ldots, \theta_N(k))$ and $S^*(k) = (S_2(k), \ldots, S_N(k))$.

By induction can be seen that

$C(k) = V(k) - \sum_{i=1}^{k}(S^*(i) - S^*(i-1))\theta^*(i),\ k = 0, 1, \ldots, T.$

The strategy generates *gains process* $\{G(k)\}$, which is the increase of the value of given process and is defined by $G(k) = (G_1(k), \ldots, G_n(k))$ and

$G(0) = 0,$

$G(k) = \sum_{i=1}^{k}(S^*(i) - S^*(i-1))\theta^*(i).$

The discounted gains process is denoted as

$$\begin{aligned}\tilde{G}(k) &= \tfrac{1}{(1+r)^k}\sum_{i=1}^{k}(S^*(i) - S^*(i-1))\theta^*(i)\\ &= \sum_{i=1}^{k}(\tilde{S}^*(i) - \tilde{S}^*(i-1))\theta^*(i) = \sum_{i=1}^{k}\Delta\tilde{S}^*(i)\theta^*(i).\end{aligned}$$

Cumulative cost can be counted as a difference of value process and gains process

$C(k) = V(k) - G(k),\ k = 0, \ldots, T,$ and

$C(k) - C(k-1) = V(k) - V(k-1) - (S^*(k) - S^*(k-1))\theta^*(k),\ k = 1, \ldots, T.$

Theorem 4.3. *If a strategy $\theta(k)$ is self-financing, then discounted value process*

$$\tilde{V}(k) = V(0) + \tilde{G}(k).$$

Proof. Fix $k \leq T$ and suppose $\theta(k)$ is a self-financing strategy. We write

$$\begin{aligned}\tilde{V}(i+1) &= \frac{V(i+1)}{(1+r)^{i+1}} = \frac{1}{(1+r)^{i+1}}(S_1(i)\theta_1(i+1) + S^*(i)\theta^*(i+1))\\ &= \frac{1}{(1+r)^{i+1}}(\theta_1(i+1) + S^*(i+1)\theta^*(i+1)) = S_1(0)\theta_1(1+i) + \tilde{S}^*(i+1)\theta^*(i+1)\end{aligned}$$

and similarly based on self-financing definition

$$\begin{aligned}\tilde{V}(i) &= \frac{V(i)}{(1+r)^{i}} = \frac{1}{(1+r)^{i}}(S_1(i)\theta_1(i+1) + S^*(i)\theta^*(i+1))\\ &= \frac{1}{(1+r)^{i}}(\theta_1(i+1) + S^*(i)\theta^*(i+1)) = S_1(0)\theta_1(1+i) + \tilde{S}^*(i)\theta^*(i+1).\end{aligned}$$

Now we have for each $i = 0, 1, \ldots, k-1$

$$\begin{aligned}\Delta\tilde{V}(i+1) &= \tilde{V}(i+1) - \tilde{V}(i) \\ &= (S_1(0)\theta_1(1+i) + \tilde{S}^*(i+1)\theta^*(i+1)) - (S_1(0)\theta_1(1+i) + \tilde{S}^*(i)\theta^*(i+1)) \\ &= \tilde{S}^*(i+1)\theta^*(i+1) - \tilde{S}^*(i)\theta^*(i+1) = \theta^*(i+1)\Delta\tilde{S}^*(i+1) = \Delta\tilde{G}(i+1)\end{aligned}$$

and now by denoting $n = i+1$

$$\tilde{V}(k) - V(0) = \sum_{n=1}^{k} \Delta\tilde{V}(n) = \sum_{n=1}^{k} \Delta\tilde{G}(n) = \tilde{G}(k).$$

□

The concept of arbitrage was introduced for one period model in Section 3.2. Now arbitrage is a bit more complicated because it is possible to perform actions at multiple times. Next we define arbitrage for multi-period model.

Definition 4.4. A self-financing trading strategy is called an *arbitrage strategy* if it's value process V satisfies the following:
i) $V(0) \leq 0$,
ii) $V(T) \geq 0$ almost surely
iii) and $\mathbb{P}(V(T) > 0) > 0$, which in this thesis setting means that $V(T, \omega) > 0$ for at least one $\omega \in \Omega$.

4.2 Fundamental theorem of arbitrage pricing

Recall Theroem 3.4. We will prove its analogue for the multi-period market model. This is one of the main theorems of the thesis.

Theorem 4.5. The first fundamental theorem of arbitrage pricing. *A market is arbitrage free if and only if there exists at least one risk neutral probability measure Q that is equivalent with the original probability measure $\mathbb{P}$, such that all discounted price processes $\tilde{S}_n(k)$, $k = 1, \ldots, T$, are martingales with respect to Q.*

Proof. Assume there exists a risk-neutral probability measure Q. Take a predictable strategy $\theta(k)$, $k = 1, \cdots, T$ and note that the discounted value process $\tilde{V}(k)$ of this strategy is a martingale by assumption. Suppose first that the value process fills the conditions of arbitrage opportunity of Definition 4.4 so that $V(0) \leq 0$, $V(k) \geq 0$ almost surely and $\mathbb{P}(V(k) > 0) > 0$. We can assume without loss of generality that $V(0) = 0$. Then by a martingale property 2.40 on $\tilde{V}(k)$ under Q

$$\mathbb{E}_Q(\tilde{V}(k)) = \mathbb{E}_Q(\tilde{V}(0)) = \mathbb{E}_Q(V(0)) = 0, \quad k = 1, \dots, T.$$

We multiply both sides by $(1+r)^k$ when we get the future value on the left and zero on the right.

$$\mathbb{E}_Q(V(k)) = \sum_{w \in \Omega} Q(w) V(k, \omega) = 0.$$

Because all terms are non-negative as $V(k)$ was defined so and surely $Q(w) > 0$ and we got $\sum_{w \in \Omega} Q(w)V(k,\omega) = 0$ for all k, must be $V(k,\omega) = 0$ for every $\omega \in \Omega$. This shows that there is no way to construct an arbitrage so the original model is arbitrage free which is an contradiction.

To prove the implication other direction we now assume the absence of arbitrage. Let the strategy $\theta(k) = (\theta_1(k), \dots, \theta_N(k))$, $k = 1, ..., T$, be predictable. The probability space $\Omega = \{\omega_1, \dots, \omega_M\}$ is finite so a random variable can be regarded as an element of Eucledian space $\mathbb{R}^M$ so now we have a discounted gains process

$$\tilde{G}(k) = (\tilde{G}(k, \omega_1), \dots, \tilde{G}(k, \omega_M)) \in \mathbb{R}^M$$

and so

$$W = \{\tilde{G}(T) : \theta(k) \text{ predictable}\} \subset \mathbb{R}^M$$

as W is clearly closed under linear combinations.

We take $V(0) = 0$ and define $\theta(k)$ to be a self-financing predictable strategy. By theorem 4.3 $\tilde{V}(T) = V(0) + \tilde{G}(T)$ and by definition 4.4 for the absence of arbitrage now $\tilde{V}(T) \leq 0$ implicating $\tilde{G}(T,\omega) \leq 0$ for all ω. Now we have shown that if $\tilde{G}(T,\omega) \geq 0$ for all ω then must be $\tilde{G}(T,\omega) = 0$ for all ω.

We apply now Lemma 2.18, which says that if $A \subset \mathbb{R}^M$ is convex and compact and W is a vector subspace in $\mathbb{R}^M$ disjoint from A, then there exists $z = (z_1, \cdots, z_M) \in \mathbb{R}^M$ such that $\langle z, a = (a_1, \cdots, a_M) \rangle > 0$ for all $a \in A$, and $\langle z, w \rangle = 0$ for all $w \in W$ where $\langle x, y \rangle$ denotes an Euclidean inner product in $\mathbb{R}^M$. We denote

$$A = \{a = (a_1, \dots, a_M) : \sum_{i=1}^{M} a_i = 1, a_i \geq 0\}.$$

Define

$$Q(w_i) = \frac{z_i}{\sum_{i=1}^{M} z_i} > 0$$

and then using the definition of inner product in $\mathbb{R}^M$ we obtain

$$\langle z, w \rangle = \sum_{i=1}^{M} z_i \tilde{G}(T, w_i) = 0$$

so

$$\sum_{i=1}^{M} Q(w_i)\tilde{G}(T, w_i) = \mathbb{E}_Q(\tilde{G}(T)) = 0.$$

But the stock holdings θ was defined as predictable process. We may define $\theta(k) = \{0, \cdots, 0, \theta_n(k), 0, \cdots, 0\}$ for $k \leq T$ and fix $n \leq d$ and then starting from the definition of discounted gains process

$$\mathbb{E}_Q(\tilde{G}(T)) = \mathbb{E}_Q\left(\sum_{i=1}^{T} \Delta\tilde{S}^*(i)\theta^*(i)\right) = \mathbb{E}_Q\left(\sum_{i=1}^{T} \Delta\tilde{S}_n(i)\theta_n(i)\right) = 0.$$

By Theorem 2.50. $\tilde{S}_n(i)$ are martingales for $n = 1, \cdots, N$, since the $\theta_n(k)$ are arbitrary predictable sequences.

□

5 Applications of arbitrage pricing

We have looked into the concept of arbitrage pricing first in one period market model in Chapter 3.1 and then multi-period market model in Chapter 4.1. We introduced some crucial theorems linked to arbitrage pricing and now we will look closer into some of the applications that arbitrage pricing system brings to the table from the view of investor and options trader. More specifically we look into a investing strategies and hedging based on expected market movements. These strategies are replicating payoffs and portfolio insurance.

Definition 5.1. Suppose there are assets A and $B_1, B_2, \cdots, B_K$ and suppose a market where these assets can be traded freely. Denote the prices for shares such that the price of asset A at time k is $S_A(k, \omega_m)$ and price of asset B_n at time k is $S_{B_n}(k, \omega_m)$ where $\omega_m = (\omega_1, \dots, \omega_M)$ is the happening scenario. We call a portfolio $\theta = (\theta_1, \dots, \theta_N)$ a replicating portfolio if

$$S_A(1, \omega_m) = \sum_{n=1}^{N} S_{B_n}(1, \omega_m)\theta_n, \quad \text{for all } m = 1, 2, \dots, M. \tag{5.1}$$

Theorem 5.2. *Suppose we have a replicating portfolio $\theta = (\theta_1, \dots, \theta_N)$ for asset A in by assets $B_1, B_2, \cdots, B_K$. If the market is arbitrage-free, then at time $k = 0$ must hold that*

$$S_A(0) = \sum_{n=1}^{N} S_{B_n}(0)\theta_n. \tag{5.2}$$

Proof. Suppose $S_A(0) \neq \sum_{n=1}^{N} S_{B_n}(0)\theta_n$. Then either $S_A(0) < \sum_{n=1}^{N} S_{B_n}(0)\theta_n$ or $S_A(0) > \sum_{n=1}^{N} S_{B_n}(0)\theta_n$. First we assume $S_A(0) < \sum_{n=1}^{N} S_{B_n}(0)\theta_n$. Now we would have a portfolio for assets $A, B_1, \cdots, B_N$ such that $\theta_* = (1, -\theta_1, \dots, -\theta_N)$. Then the value at time $k = 0$ would be

$$S_A(0) - \sum_{n=1}^{N} S_{B_n}(0)\theta_n < 0$$

but it's value et time $k = 1$ would be

$$S_A(1, \omega_m) - \sum_{n=1}^{N} S_{B_n}(1, \omega_m)\theta_n \leq 0,$$

which would lead to an arbitrage based on arbitrage definition 3.2.

Then we assume $S_A(0) > \sum_{n=1}^{N} S_{B_n}(0)\theta_n$. Now we would have a portfolio for assets $A, B_1, \ldots, B_N$ such that $\theta_* = (-1, \theta_1, \ldots, \theta_N)$. Then the value at time $k = 0$ would be

$$\sum_{n=1}^{N} S_{B_n}(0)\theta_n - S_A(0) < 0$$

but it's value et time $k = 1$ would be

$$\sum_{n=1}^{N} S_{B_n}(1, \omega_m)\theta_n + S_A(1, \omega_m) \leq 0$$

which would be an arbitrage based on arbitrage Definition 3.2. □

A self financing portfolio Definition 4.1 is a replicating portfolio. Replicating portfolios can be used for hedging in financial markets. In following sections we see some examples that can be replicated.

5.1 Options strategies

A *butterfly spread* is an options strategy. It is designed to be used in a situation where the price of the portfolio doesn't change much. Butterfly spread offers a good probability to earn a limited profit with a fixed risk. Butterfly spread option strategy involves a combination of bull spreads and bear spreads. *Bull spread* is a bullish option trading strategy. Investors expectation towards the market is said to be *bullish* when one thinks the prices will rise in the stock market. Bull spread involves buying and selling either call options or put options with different strike prices of the same underlying asset and the same expiration date. The option with the lower strike price is bought and the option with the higher strike price is sold. A bull spread is an optimistic options strategy designed to work in a slightly rising markets. Bull spreads achieve maximum profit if the underlying asset closes at or above the higher strike price. *Bear spread* is a bearish option trading strategy. In contrast to bullish market sentiment *bearish* attitude means that investor expects a slight downward price movement in the stock market. Like bull spread a bear spread can be constructed by using either put options or call options. For more, see example in reference [15].

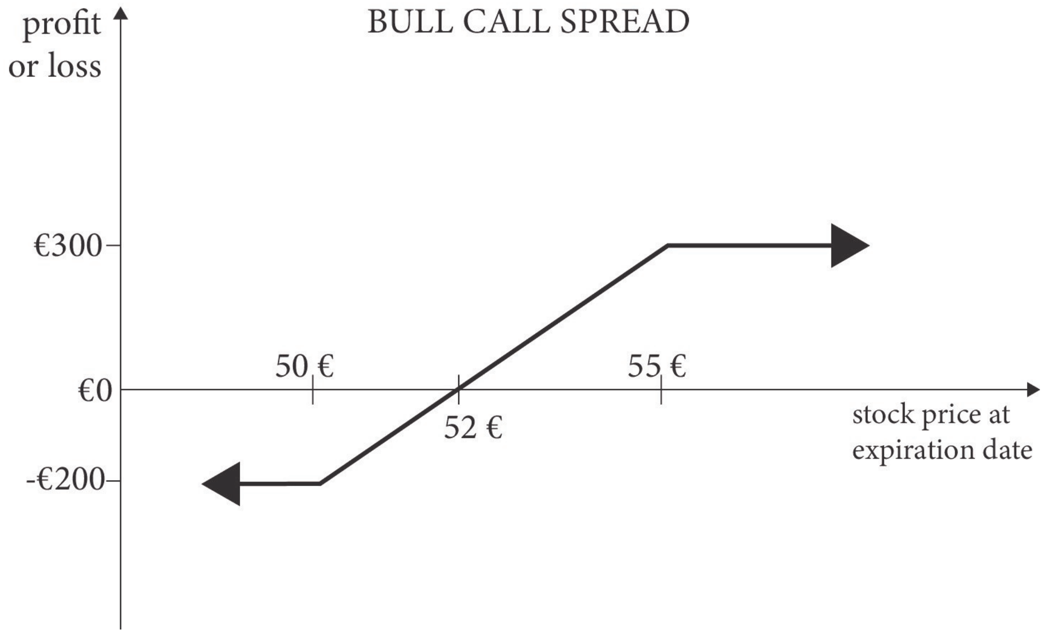
Butterfly spreads use four option contracts with the same expiration date but three different strike prices. There is multiple ways to construct the butterfly spreads but they all have these three strike prices: a higher strike price, an at-the-money strike price and a lower strike price. *At-the-money* strike price, ATM, means that options strike price is identical to the price of the underlying security. The higher strike price and the lower strike prices are equidistant from the at-the-money options price. For example if the price of the underlying security is 100 euros, the at-the-money price is also 100 euros. Then lower options strike price could be 90 euros and upper options strike price 110 euros when both have the same 10 euro difference to the middle price. Combining call or put options in different ways butterfly spreads are designed to either profit from volatility or low volatility. Volatility is a statistical measure that tells us how the return is spread over the different securities. It tells on how wide range an assets prices can be around mean price [19]. We also introduce and *neutral calendar spread* and an *iron condor spread* which are options trading strategies similar to butterfly spreads.

A *bull call spread* option trading strategy is used when the options trader thinks that the price of the underlying asset will go moderately up in the near future. A bull call spreads can be constructed by buying an at-the-money call option and at the same time selling a higher striking out-of-the-money call option of the same underlying security and the same expiration date. A call option is said to be out-of-the-money (OTM) if the underlying assets price is below the strike price. Next we will introduce an example of bull call spread taken from reference [20].

Example 5.3. An options trader believes that the price of the stock X trading at the moment at 52 euros is going to rise soon and enters a bull call spread by buying a July 50 call for 300 euros and selling a July 55 call for 100 euros. By July 50 we mean a call option that gives the option owner the right but not obligation to buy the the stock for a price 50 euros with expiration in July. The net investment required from investor to enter this spread is a debit of 200 euros.

The price of the stock X begins to rise and closes at 56 euros on expiration date. Both options expire in-the-money with the July 50 call having an intrinsic value, meaning the amount of profit that exists in an options contract [15], of 600 euros and the July 55 call having an intrinsic value of 100 euros. A call option is said to be in-the-money (ITM) if the market price is above the strike price. This means that the spread is now worth 500 euros at expiration. Since the trader had a debit of 200 euros when he bought the spread, his net profit is 500 euros $-$ 200 euros $=$ 300 euros.

If the price of X had dropped to 48 euros instead, both options would have expired worthless. The investor would lose his entire investment of 200 euros, which is also the maximum possible loss. Below is pictured this bull call spreads profit/loss as a function of stock price at expiration date.

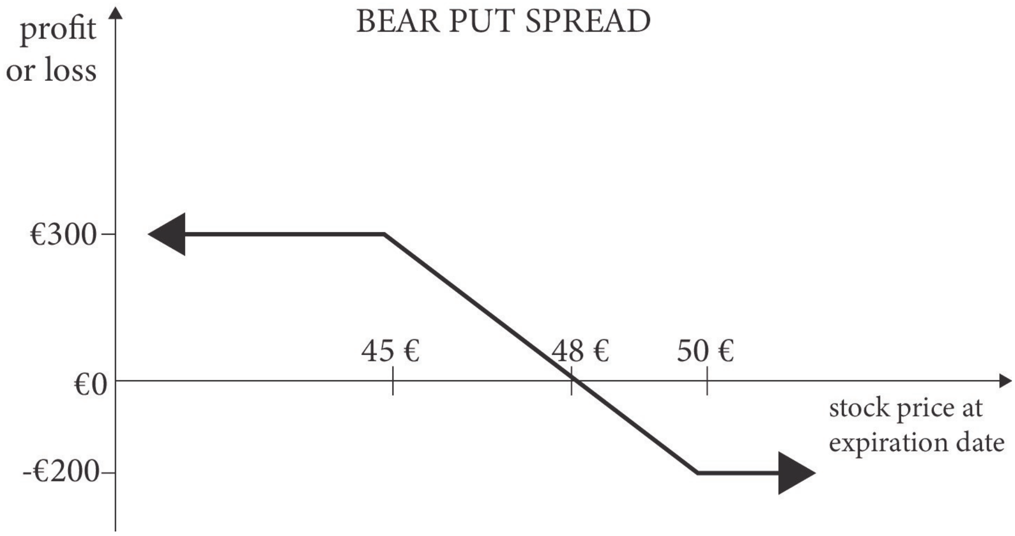


A *bear put spread* option trading strategy is used when the options trader thinks that the price of the underlying asset will go moderately down in the near future. A bear put spread is constructed by buying a higher striking in-the-money put option and selling a lower striking out-of-the-money put option of the same underlying security with the same expiration date. A put option is said to be in-the-money (ITM) if the market price is below the strike price and out-of-the-money if the underlying assets price is above the strike price. Next we introduce an example of a bear put spread taken from reference [20].

Example 5.4. Suppose the price of the stock X is at 48 euros in June. An options trader believes that the price of the stock X will go downwards soon and decides to enter a bear put spread position by buying a July 50 put for 300 euros and selling a July 45 put for 100 euros at the same time. This means a net debit of 300 euros $-$ 100 euros $=$ 200 euros for entering this position. Similarly to Example 5.3 July 45 put gives the right but not obligation for the option owner to sell stock for price 45 euros and expires in July.

The price of X stock drops to 44 euros at expiration. Both puts expire in-the-money with the July 50 call bought having 600 euros in intrinsic value and the July 45 call sold having 100 euros in intrinsic value. Since the trader took a debit of 200 euros and sold July 40 for 100 euros the total net profit is 600 euros $-$ 100 euros $-$ 200 euros $=$ 300 euros. This is also the maximum possible profit.

If instead the stock price had rise to 52 euros would both options expire worthless. Then the options trader would lose the entire debit of 200 taken to enter the trade. This is also the maximum possible loss. Below pictured this bear put spreads profit/loss as a function of stock price at expiration date.



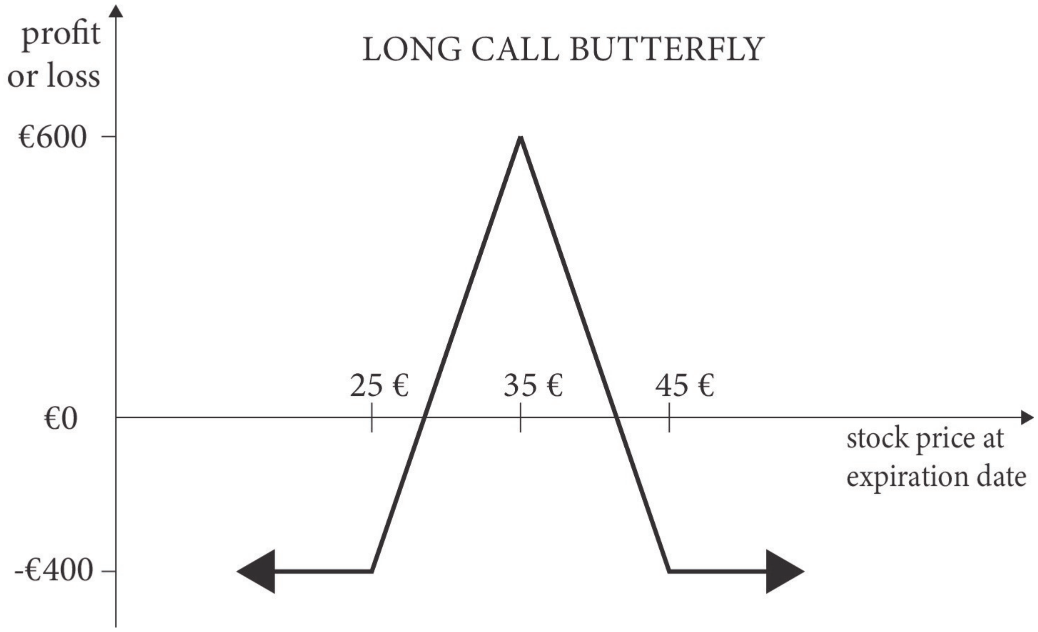
Like said before a butterfly spread is a combination of a bear spread and a bull spread. There is different kind of butterfly spreads but the common key factor in all butterfly spread strategies is the expected low volatility in the near future. This means that the investor thinks that the underlying stock will not rise or fall much by expiration date. Next we introduce two different kind of butterfly spreads: a *long call butterfly* and a *short put butterfly* and two other options strategies an *iron condor* and a *neutral calendar spread.*

First we look into the long call butterfly spread. The long call butterfly spread can be constructed by buying one lower striking in-the-money call, selling two at-the-money calls and buying another higher striking out-of-the-money call. Maximum profit for the long butterfly spread is achieved when the underlying stock price remains unchanged at expiration. Only the lower striking call option expires in-the-money at this price. Maximum loss for the long butterfly spread is limited to the debit taken to enter the trade plus commission fees. Next we introduce an example of a long butterfly call spread taken from reference [20].

Example 5.5. Suppose a stock X has a price 35 euros in June. An options trader enters a long call butterfly by purchasing a July 25 call for 1100 euros, selling two July 35 calls for 400 euros each and purchasing another July 45 call for 100 euros. The net debit options trader takes to enter the position is 1100 euros $-$ 400 euros $-$ 400 euros $+$ 100 euros $=$ 400 euros. This is also the maximum possible loss of this butterfly spread.

On expiration in July the stock X is still trading at the same 35 euros. The July 35 calls and the July 45 call expire worthless while the July 25 call still has an intrinsic value of 1000 euros. Subtracting the initial debit of 400 euros, the resulting profit is 1000 euros $-$ 400 euros $=$ 600. This is also the maximum possible profit.

Maximum loss of this butterfly spread results when the stock is trading below 25 or above 45. All the options expires worthless at 25 wuros price. When the price of the stock X goes above 45 euros, any profit achieved from the two long calls will be neutralised by the loss from the two short calls. In both situations, the butterfly trader suffers maximum loss which is the initial debit taken to enter the trade 400 euros. Below pictured this long call butterfly spread put spreads profit/loss as a function of stock price at expiration date.

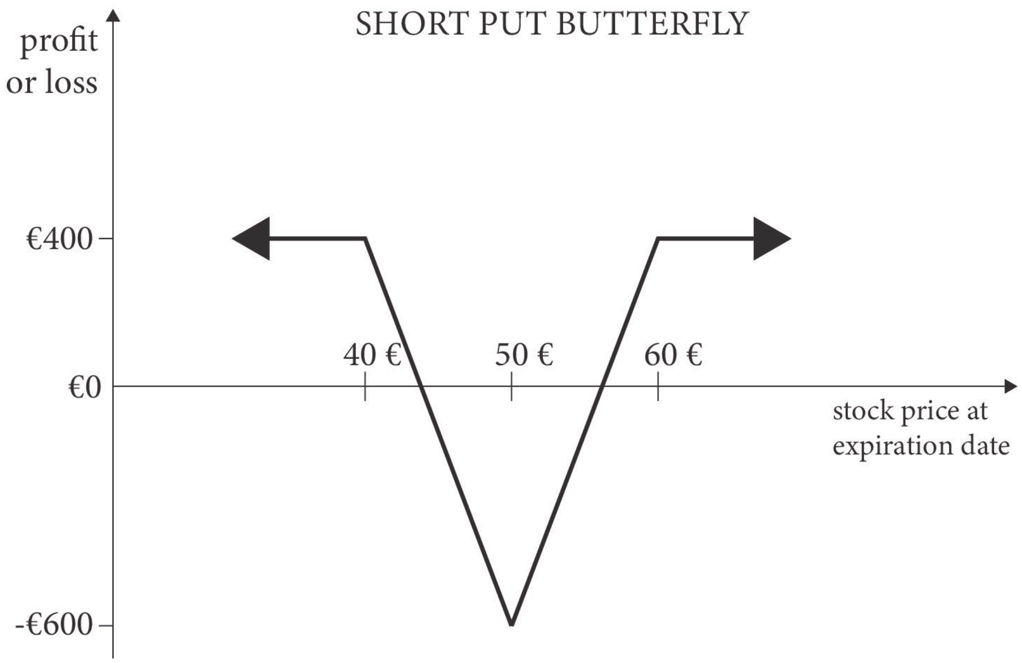


A *short put butterfly* options strategy is similar to long call butterfly but used when the options trader expects the stock price to change. As other butterfly spreads this options strategy has a limited profit and limited loss. Short put butterfly is constructed by selling one lower striking out-of-the-money put, buying two at-the-money puts and selling one higher striking in-the-money put. Maximum profit will be achieved when the stock price rises above the higher strike price or drops below the lower strike price at the expiration date. If the price of the stock remains unchanged options trader will face the maximum loss. At this price the higher striking put expires in the money and options trader needs to buy it back to exit the trade. Next we introduce an example of a short butterfly put spread taken from reference [20].

Example 5.6. Suppose the price of a stock X is at 50 euros in June. To enter a short put butterfly an options trader by sells a July 40 put for 100 euros, buys two July 50 puts for 400 each and sells a July 60 put for 1100 euros. The total credit taken to take this position is 100 euros $-$ 400 euros $-$ 400 euros $+$ 1100 euros $=$ 400 euros, which is also the maximum possible profit.

The stock price drops to 40 euros by expiration date in July. Now the options expire worthless and the short put butterfly trader gets to keep the entire initial credit taken of 400 as profit. This is the maximum profit attainable and it would be same if the stock had rise up to 60 euros instead or beyond.

Maximum loss would happen if the stock price would instead remain at 40 euros at expiration. At this price, all puts would expire worthless except the higher striking put. The higher striking put sold short would have a value of 1000 euros and the options trader has to buy it back to close the trade. The maximum loss would now be 1000 euros − 400 euros = 600 euros. Below pictured this short put butterfly spread put spreads profit/loss as a function of stock price at expiration date.

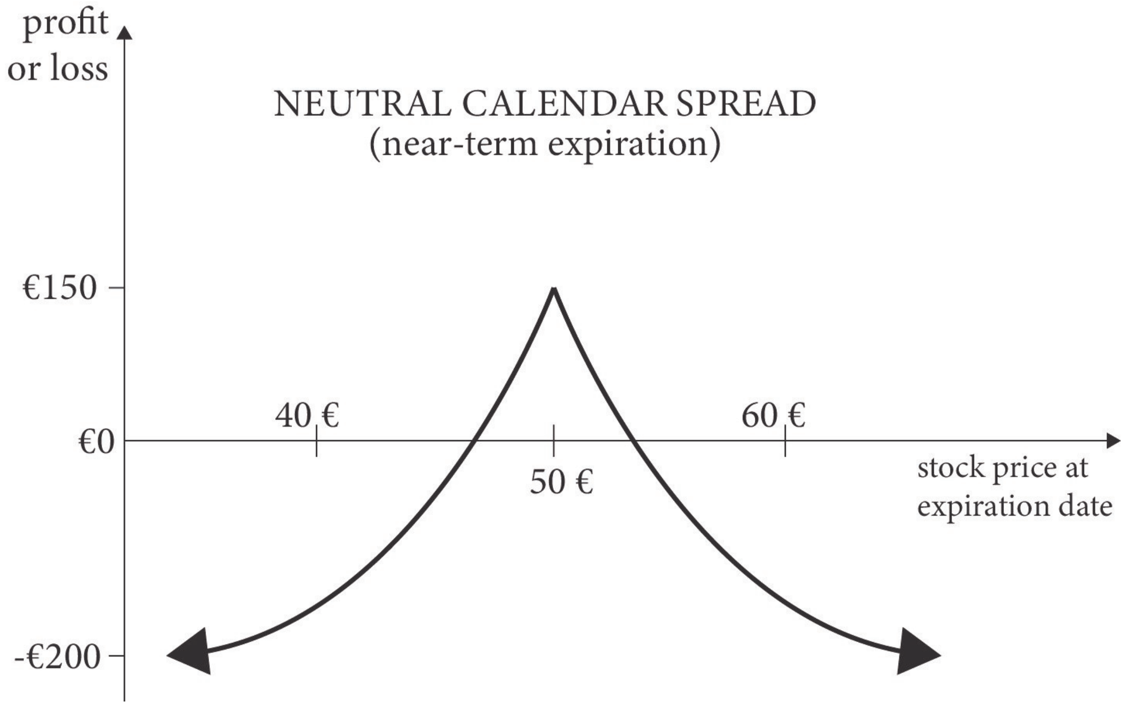


A *neutral calendar spread* options strategy is used when the options trader expects the stock price to stay relatively same for a period of time. Calendar spreads minimizes risk by buying the same amount of options of the same underlying security with the same strike price and different expiration dates. Neutral calendar spread is constructed by buying long term calls and selling the same amount of near term at-the-money calls. Maximum profit is the amount received from the sale of the near term options minus any time decay of the longer term options. If the price of the stock goes down and stays down until expiration of the longer term options the options trader will face the maximum loss which is limited to the net debit taken to enter the position. Next we introduce an example of a neutral calendar put spread taken from reference [20].

Example 5.7. Suppose the price of a stock X is at 50 euros in June. An options trader believes the stock price will go slightly up and slightly up in the next few months. To enter a neutral calendar spread an options trader sells a July 40 call for 200 euros and buys a October 40 call for 400. The total debit taken to take this position is 400 euros $-$ 200 euros $=$ 200 euros, which is also the maximum possible loss.

The stock price stays at 50 euros by expiration date in July and the near term call expires worthless. The long term call also lost some value because of the time decay but is still worth 350 euros. Now the trader can sell this option and will receive a total profit of the price of the long term call minus the net debit taken 350 euros $-$ 200 euros $=$ 150 euros.

Maximum possible loss would occur if the price of the stock X had instead declined to 47 euros and stayed at 47 euros until October. Then both options would have expired worthless. he would lose his entire investment of 200 euros.



An *iron condor* options strategy is used when the options trader expects the stock price to have low volatility. Iron condor gives the trader a large probability to earn a low profit. Iron condor is basically a combination of a bull put spread and a bear call spread. Iron condor is constructed with 2 options by selling a lower strike out-of-the-money put, buying another out-of-the-money put with even lower strike price, selling out-of-the money call for a higher strike price and buying another out-of-the-money call with even higher strike price.

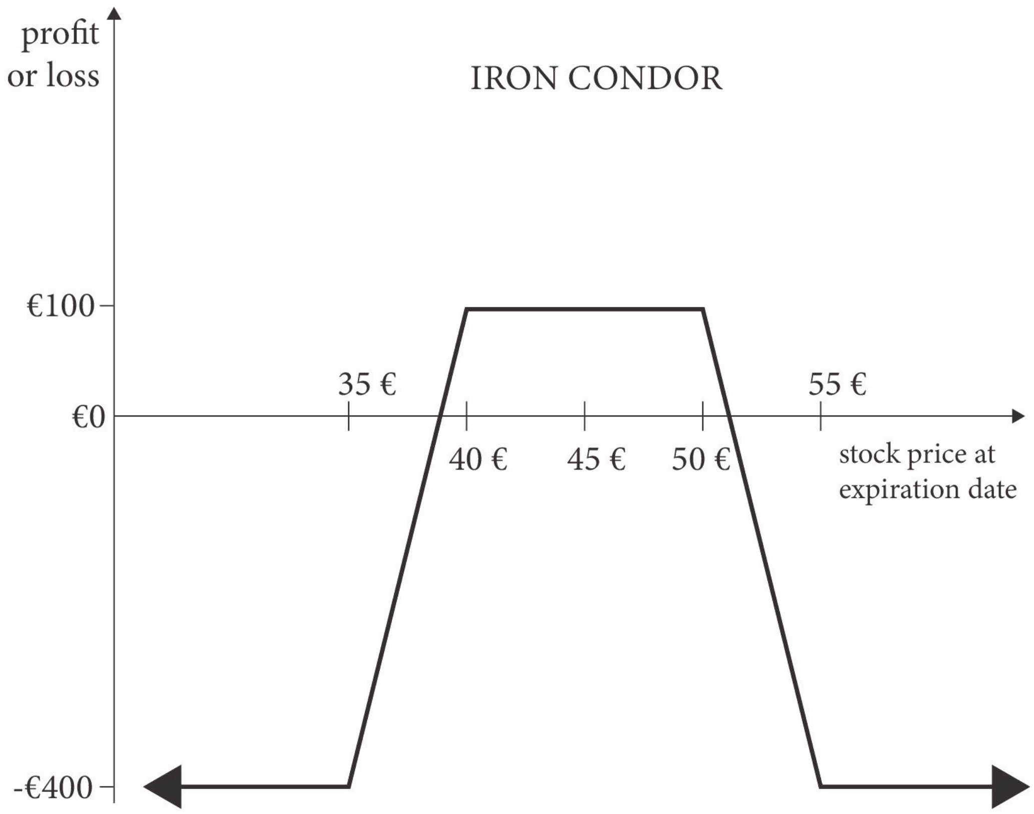
Maximum profit is limited to the amount of net credit to enter iron condor and will be received if the price of the stock is between the strike prices of the sold call and sold put at expiration. Also the maximum loss is limited but it is much higher than the maximum profit. The idea of iron condor is that it is not very likely the maximum loss to happen. It occurs when the stock price falls at or below the lower strike of the put option bought or rises above or is equal to the higher strike of the call option bought. Maximum possible loss is the difference in between the strike prices of calls (or puts) minus the net credit received. Next we introduce an example of an iron condor put spread taken from reference [20].

Example 5.8. Suppose the price of a stock X is at 45 euros in June. An options trader believes the stock price will move in the next few months but not too far from Junes strike price. To enter an iron condor spread an options trader sells a July 50 call for 100 euros to buy a July 55 call for 50 and sells a July 40 put for 100 to buy a July 35 for 50 euros. The total net debit received by taking this position is 100 euros + 100 euros − 50 euros − 50 euros = 100 euros, which is also the maximum possible profit.

The stock price stays at 45 euros by expiration date in July and all the four options expire worthless. Now the trader can keep the whole net credit 100 euros.

If the price would fall to 35 euros all the options except July 40 call would expire worthless. The put July 40 has an intrinsic value of 500 euros. The options trader has to buy this option back to exit the trade. Now the maximum possible loss would occur and with the initial 100 credit received it is 500 euros − 100 euros = 400 euros. The maximum loss situation would also happen if the stock price had gone up to 55 euros instead.

If the price would decline under 35 euros, for example to 30 euros on expiration both the July 35 put and the July 40 put options would expire in-the-money. The July 35 put has an intrinsic value of 500 euros while the July 40 put is worth 1000 euros. The trader can sell the July 35 put for 500 euros but he has to buy July 40 put back for 1000 euros to exit the trade. Taking in account the the initial credit of 100 euros received the maximum possible loss is $1000 - 500 - 100 = 400$ euros. The same would happen if the stock price would go above 55 euros instead.



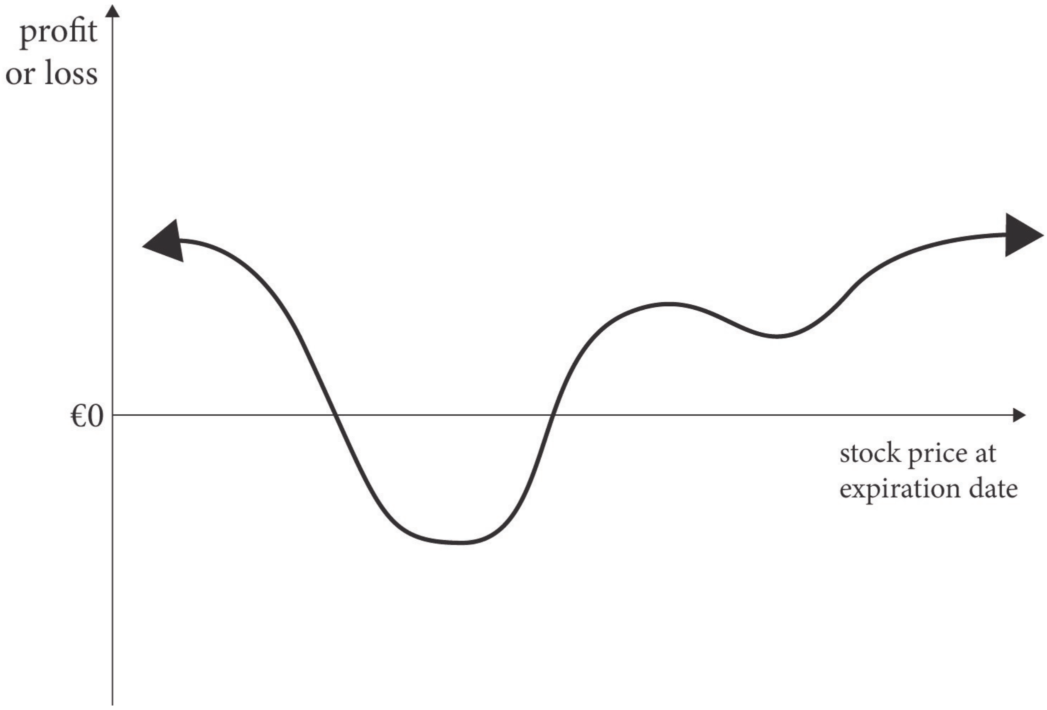
The payoffs of all trading strategies can be replicated. Next we introduce the calculation formula for the payoff of a butterfly spread constructed by combining call and put options.

Example 5.9. Recall the payoff forming of a put and call options Examples 2.4 and 2.3. Suppose an agent has an option to sell or buy an asset at time 1 for a fixed strike price K specified at time 0. Recall price at time 1 equation 3.2 and denote $V(1) = \theta \cdot S(1)$ price of a given portfolio at time 1. Denote $V(0)$ price of a given portfolio at time 0 equally to Definition 2.45 $V(0) = \mathbb{E}_Q\left[\frac{V(1)}{1+r}\right]$. The payoff off a butterfly spread denoted by C is of the form

$$C = (K- \mid V(1) - V(0) \mid)^+.$$

Clearly the payoff of the butterfly spread is maximal when $V(1) = V(0)$ and decreases if the price at time 1 of the portfolio θ deviates from its price at time 0 as it should be as a butterfly spread option is used when the price is expected to stay close to its present value. By denoting $K_\pm = V(0) \pm K$ we can represent C as combinations of call or put options on $V(1)$:

$$\begin{aligned} C &= (V - K_-)^+ - 2(V(1) - V(0))^+ + (V - K_+)^+ \\ &= -(K_- - V)^+ + 2(V(0) - V(1))^+ - (K_+ - V)^+. \end{aligned}$$



5.2 Portfolio insurance

Portfolio insurance is a hedging technique used when investors are not certain of the direction of the market and they want to protect their assets. In this technique the goal is to maintain the portfolio value by buying and selling securities periodically. The idea behind the portfolio insurance strategy is to combine financial instruments so that it protects the investor against risk. The technique limits loss but it also limits any gains. The portfolio insurance strategy is executed by buying index put options. It can alternatively be done by using listed index options [15].

Portfolio insurance is designed to react to market movements by selling in a falling market and buying in a rising market. Portfolio insurance is designed to make it possible for investors to participate in a rising market but still protect their portfolio if the market suddenly falls. In practice a portfolio insurance uses computer-based models that are based on stock options analysis and with this information portfolio insurance agents compute optimal price ratios at current stock market conditions. Usually portfolio insurance investors adjusts to moving market situation by trading index futures instead of buying and selling stocks. A portfolio insurer sells index futures when market drops and buys index futures when stock values rise. The number of owned stocks stays the same but the portfolio value changes. More detailed explanation and examples of situations of portfolio insurance used in real life situation in reference [16].

We will be looking into a portfolio insurance example in one period market model setting defined in Section 3.1 taken from reference [1].

Example 5.10. In portfolio insurance the main idea is to increase exposure to rising assets and to reduce to falling prices. This suggests to replace the payoff $V = \theta \cdot S(1)$ of a given portfolio by a modified portfolio denoted by $h(V)$, where h is a convex and increasing function. First we consider a case where $V \geq 0$. When this holds, the corresponding payoff $h(V)$ can be expressed as a combination of investments in bonds, in V itself, and in basket call options on V. Recall Proposition 2.20 part d equation 2.1. Here convexity implies that $h(x) = h(0) + \int_0^x h'(y)dy$ for the for the increasing right-hand derivative $h' := h'_+$ of h. Note that h' can be presented as the distribution function of a positive Radon measure γ on $[0, \infty) : h'(x) = \gamma([0, x])$ for $x \geq 0$. Hence, Fubinis theorem 2.28 equation (2.2) implies that

$$\begin{aligned} h(x) &= h(0) + \int_0^x \int_{[0,y]} \gamma(dz)dy \\ &= h(0) + \gamma(\{0\})x + \int_{(0,\infty)} \int_{\{y|z \leq y \leq x\}} dy\gamma(dz). \end{aligned}$$

Since the inner integral equals $(x - z)^+$, defined as in Example 2.44, we obtain

$$h(V) = h(0) + h'(0)V + \int_{(0,\infty)} (V - z)^+ \gamma(dz).$$

The payoff $V = \theta \cdot S$ may take negative values if the portfolio θ contains also short positions. In this case, the increasing convex function h must be defined on all of $\mathbb{R}$. It's right-hand derivative h' can be represented as

$$h'(y) = h(0) + h'(x) = \gamma((x, y]), \quad x < y,$$

for a positive Radon measure γ on $\mathbb{R}$. Looking seperately at the cases $x < 0$ and $x \geq 0$, we see that

$$h(x) = h(0) + h'(0)x + \int_{(0,\infty)} (x - z)^+ \gamma(dz) + \int_{(-\infty,0]} (z - x)^+ \gamma(dz).$$

Thus, the payoff $h(V)$ can be realized by holding bonds, shares in V, and a mixture of call and put options on V :

$$h(V) = h(0) + h'(0)V + \int_{(0,\infty)} (V - z)^+ \gamma(dz) + \int_{(-\infty,0]} (z - V)^+ \gamma(dz).$$

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